Lecture Notes
CS131

Part 1

General

1. Two's complement: $x \bmod 2^N$

2. Decimal to Binary: keep halving the number, noting remainders. Take digits from end to start.

3. Division algorithm: $a, b \in \mathbb{Z}, b \neq 0 \to \exists$ unique $q, r \in \mathbb{Z}$ (quotient, remainder) s.t. $a = qb + r, 0 \leq r < |b|$

4. $b \in \mathbb{Z}$ divides $a \in \mathbb{Z}$ if $a = qb$ for some $q \in \mathbb{Z}$.

5. $\gcd(0, n) = n \forall n > 0$

6. **Congruence** $a \equiv b \pmod{n}$ if $a = b + kn, k \in \mathbb{Z}$

7. Every rational number has a **co-prime** number with $\gcd(m, n) = 1, n \geq 1$ and inverse $(q \cdot q^{-1} = 1)$

8. **Algebraic** number $n \in \mathbb{R}$ if solution of a polynomial equation with rational coefficients. Otherwise called **Transcendental**

9. $U \in \mathbb{R}$ is the least upper bound (**supremum**) of S if U is an upper bound of S and $U \leq u$ for every upper bound u of S.

10. $L \in \mathbb{R}$ is the greatest lower bound (**infimum**) of S if L is a lower bound of S and $L \geq l$ for every lower bound l of S

11. Complex numbers $\mathbb{C}$ in form of $a + ib, a, b \in \mathbb{R}, i^2 = -1$ It holds: $a + ib = c + id \Leftrightarrow a = c, b = d$, separating into real (a, c) and imaginary (b, d) parts. i isn't in the imaginary part! Complex **conjugate**: $\overline{a + ib} = a - ib$ The real part of $z \in \mathbb{C}$ is $\frac{(z+\bar{z})}{2}$, imaginary part is $\frac{(z-\bar{z})}{2i}$

12. **Polar** Coordinates: $x + iy = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$ is the **modulus** of $x + iy$, denoted as $|x + iy|$, representing distance between coordinate and origin, and $\tan\theta = \frac{y}{x}$, called the **argument**, with principal argument satisfying $-\pi < \theta \leq \pi$

Theorems and Important

13 Axioms of $\mathbb{R}$
1. Commutativity $x + y = y + x$
2. Associativity $x + (y + z) = (x + y) + z$
3. Distributivity $x.(y + z) = x.y + x.z$
4. Additive ident. $\exists 0 | x + 0 = x$
5. Multiplicative id. $\exists 1 | x.1 = x$
6. id's 4,5 are unique $1 \neq 0$
7. Every $n \neq 0 \in \mathbb{Z}$ has additive inverse: $x + (-x) = 0$
8. Every $n \neq 0 \in \mathbb{Z}$ has multipl. inverse: $x.x^{-1} = 1$
9. Transitivity: $x, y \wedge y < z \to x < z$
10. Trichotomy law: $x < y$ or $y < x$ or $x = y$
11. Preserv. ordering under add. $x < y \to x + a < y + a$
12. Pres. ordering under mult. $a > 0 \wedge x < y \to x.a < y.a$
13. Completeness: Every non-empty subset that bounded above has a least upper bound

Properties of the modulus. *For any $z, w \in \mathbb{C}$:*

1. $|z| = |\bar{z}|$,
2. $|z| = \sqrt{z\bar{z}}$,
3. $z\bar{z} = |z|^2$,
4. $|zw| = |z||w|$,
5. $|z + w| \leq |z| + |w|$ (the triangle inequality),
6. $||z| - |w|| \leq |z - w|$.

Theorem $\nexists x \in \mathbb{Q}$ s.t. $x^2 = 2$.
Assume opposite, show that $x = \frac{a}{b}$ with $\gcd(a, b) = 1$, then show that a, b even hence contradicting $\gcd = 1$

De Moivre's Theorem For any integer n,
$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

The Archimedean Property of the Reals
Given any $\epsilon \in \mathbb{R}^+$ there exists $n \in \mathbb{N}$ such that $n\epsilon > 1$

Fundamental Theorem of Algebra Every polynomial equation of degree n with complex coefficient has exactly n solutions in $\mathbb{C}$

Euclidean Algorithm: $\gcd(m, n)$: For $i = 1, 2, 3..$
if $r_i = 0$: output r_{i-1}; if $r_i \neq 0$, divide r_{i-1} by r_i and let r_{i+1} be the remainder.

Vectors

1. $\mathbb{R}^2 = \{(x,y)|x,y \in \mathbb{R}\}$ (Just points in a plane)
2. $\underline{a} + \underline{b} = (a_1 + b_1, a_2 + b_2)$ and $\lambda\underline{a} = (\lambda a_1, \lambda a_2)$
3. $\overrightarrow{OP} \equiv \underline{p} = (p_1, p_2) \in \mathbb{R}^2$ where O-origin, P-(p_1, p_2). Vector $\underline{p}$ is called **position vector** of point P. Two vectors are equivalent if they have same length and direction. Given A, B have position vectors $\underline{a}, \underline{b}$, then $\overrightarrow{AB} = \underline{b} - \underline{a}$.
4. Length $|\underline{a}| = \sqrt{a_1^2 + a_2^2}$. **Unit vector** length is 1. Distance between $\underline{a}$ and $\underline{b} = |\underline{b} - \underline{a}|$. To find unit vector $\underline{u}$, parallel to $\underline{v}$, use $\underline{u} = \frac{\underline{v}}{|\underline{v}|}$
5. Scalar(dot) Product: $\underline{a}.\underline{b} = a_1b_1 + a_2b_2 = |\underline{a}||\underline{b}|\cos\theta$. Vectors are orthogonal(perpendicular) if their scalar product is 0 and parallel if 1.

Linear Combinations

1. If $\underline{u}, \underline{v} \in \mathbb{R}^2, \alpha, \beta \in \mathbb{R}$, then vector of form $\alpha\underline{u} + \beta\underline{v}$ is a **linear combination** of $\underline{u}, \underline{v}$. $(6,6) = 1\cdot(0,3) + 3\cdot(2,1)$. If $\underline{u}, \underline{v}$ non-parallel, then linear combination represents a **diagonal** of a parallelogram. Linear combination with itself is called **scaling**: $2 \cdot \underline{v}$.
2. **Span** of $U = \{\alpha_1\underline{u}_1 + ... + \alpha_m\underline{u}_m | \alpha_1, ...\alpha_m \in \mathbb{R}\}$ (set of all linear combinations of its elements). Span of $\{(1,0),(0,1)\} = \mathbb{R}^2$. If one of the components is 0 in both $\underline{u}, \underline{v}$, then any vector with same component being 0 is a linear combination of $\underline{u}, \underline{v}$: $(1,0,4), (8,0,5)$.
3. **Subspace** of $\mathbb{R}^n$ is non-empty $S \subseteq \mathbb{R}^n$ with
$1)\underline{u}, \underline{v} \in S \rightarrow \underline{u} + \underline{v} \in S, \quad 2)\underline{u} \in S, \alpha \in \mathbb{R} \rightarrow \alpha\underline{u} \in S$. Closure under addition and scalar multiplication. Every subspace of $\mathbb{R}^n$ contains a zero vector. If Nonempty finite $U \subseteq \mathbb{R}^n$, then span of U is subspace of $\mathbb{R}^n$, called subspace spanned (generated) by U.

Linear Independence

1. Set $\{\underline{u_1}, ...\underline{u_m}\} \subseteq \mathbb{R}^n$ is **linearly dependent** if $\exists\alpha_1, ...\alpha_m \in \mathbb{R}$ not all zero s.t. $\alpha_1\underline{u}_1 + ... + \alpha_m\underline{u}_m = \underline{0}$, linearly independent otherwise. Any set containing $\underline{0}$ is linearly dependent. Set S is linearly dependent if one of the vectors is a linear combination of other vectors in S. $\downarrow$
2. **Predecessor Theorem**: set $\underline{u}_1, ...\underline{u}_m$ of nonzero vectors is linearly dependent iff some $\underline{u}_r$ is a linear combination of its predecessors $\underline{u}_1, ...\underline{u}_{r-1}$. **UNDERSTAND THE PROOF**

Basis and Dimension

1. Let S be subspace of $\mathbb{R}^n$, then a set of vectors is called a **basis** if it's linearly independent and spans S: $\{(0,1),(1,0)\}$ is basis for $\mathbb{R}^2$, moreover, if it's 1, with everything else 0 like above, it's a **standard basis**.
2. **Theorem**: Let S be subspace of $\mathbb{R}^n$, if set $\{\underline{u}_1, ..., \underline{u}_m\}$ spans S then any linearly independent subset of S contains at most m vectors. **UNDERSTAND THE PROOF**
3. **Dimension** of subspace of $\mathbb{R}^n$ is the number of vectors in a basis for the subspace. Any two bases for a subspace S have the same number of elements.
4. Let $\{\underline{v}_1, ...\underline{v}_m\}$ be set of nonzero vectors that spans m-dimentional subspace S of $\mathbb{R}^n$. Then removing each linear combination of its predecessors $\underline{v}_i$ will leave a basis for S. The basis will have **exactly** m vectors, any subset of S with $> m$ vectors is linearly dependent.

Matrix Algebra

1. A matrix A of order $m \times n$ is an array of numbers arranged in m rows and n columns and usually written as
$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \quad \text{or} \quad A = [a_{ij}]_{m \times n}.$$
2. **Column** matrix: $m \times 1$, **row** matrix or row vector: $1 \times n$. **Zero** matrix $O_{m\times n}$ - all elements are 0. **Negative** of A is matrix $-A = [-aij]_{m\times n}$. **Square** matrix: $m = n$. **Diagonal** matrix diag$[a_{11}, ..., a_{nn}]$: only elements $a_{11}, ...a_{nn}$ are non-zero. **Identity** matrix is square diagonal matrix with 1's as entries.
3. $A + B = [a_{ij} + b_{ij}]_{m\times n}$, only defined for **same order** matrices. Scalar multiplication $\lambda A = [\lambda a_{ij}]_{m\times n}$. Nothing special in properties of these operations.
4. Multiplication $A = [a_{ij}]_{x\times y}, B = [b_{ij}]_{y\times z}.AB = [c_{ij}]_{x\times b}$ Multiply **rows** of left matrix **by columns** of the right one. Matrix multiplication **isn't commutative**: $AB \neq BA$
5. Properties of matrix multiplication/addition:
1. $IA = A = AI$ 2. $OA = O = AO$
3. $A^xA^y = A^yA^x$ 4. $A + (-A) = O$

Matrix Determinants and Inverse

1. **Transpose** A^T is matrix A with swapped rows and columns, so $([a_{ij}]_{m\times n})^T = [a_{ji}]_{n\times m}$. Properties:
   1. $(A^T)^T = A$ 2. $(A+B)^T = A^T + B^T$
   3. $(\lambda A)^T = \lambda A^T$ 4. $(AB)^T = B^T A^T$
2. **Inverse**: If A, B are square, have same order, then an inverse of A, denoted A^{-1} is B if $AB = I = BA$. An inverse is unique. $A^{-1} = B \to B^{-1} = A$.
3. The **Determinant** of 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $(ad - bc)$ and is denoted by $\det(A)$ or $|A|$. A 2×2 matrix A is invertible iff its determinant is nonzero. $|A| = |A^T|, |AB| = |A||B|$.
4. System of **linear equations** can be written as $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \equiv \begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$
   Can (1) interchange two rows; (2) multiply a row by a nonzero number; (3) add a multiple of one row to another in the **augmented** matrix $\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$.
5. Matrices $A \sim B$ are **row equivalent** if A can be transformed into B using finite number of elementary row operations. **Row echelon** form: $\begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 \\ 0 & 0 & x_6 \end{bmatrix}$. **Elementary** matrix E is obtained from I by applying basic row operations, and is used for matrix transformations. (1)$E_{ij}(-\mu)E_{ij}(\mu) = I$, (2)$E_{ij}E_{ji} = I$; and (3)$E_i(\frac{1}{\lambda})E_i(\lambda) = I$, all of which are commutative.
6. 3×3 determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$: choose a row, mutliply by det of 4 numbers *excluded* by same row and column: $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$
   Remember about alternating checkerboard signs!
7. If $A = [a_{ij}]$ is $n\times n$ matrix then ij^{th} **minor** M_{ij} of A is determinant of $(n-1)\times(n-1)$ matrix obtained by deleting i^{th} row and j^{th} column from A. The ij^{th} **cofactor** $A_{ij} = (-1)^{i+j}M_{ij}$ (minor with alternating signs). Determinants are recursive with base case 1×1.
8. If matrix B obtained from A: (1) multipy a row of A by number $\lambda \to |B| = \lambda|A|$, (2) Interchange two *rows* of $A \to |B| = -|A|$, (3) Add a multiple of one row of A to another $\to |B| = |A|$.
9. **Adjoint** matrix is the transpose of a matrix of cofactors of A. If $|A| \neq 0$ then it's invertible, and the matrix **inverse** $A^{-1} = \frac{1}{|A|}\text{adj}(A)$
10. Set of n vectors in $\mathbb{R}^n$ is linearly independent (therefore a basis) iff it is the set of **column** vectors of a matrix with nonzero determinant.

Linear Transformations

1. Function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a **linear transformation** if $\forall \underline{u}, \underline{v} \in \mathbb{R}^m, \lambda \in \mathbb{R} : T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$ and $T(\lambda\underline{u}) = \lambda T(\underline{u})$ (preservation under addition and scalar mutliplication)
2. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transofrmation, then $T(\underline{0}) = \underline{0}$. In the exam, try to substitute 0, and check whether the output is also 0 when determining whether it's a linear transformation.
3. Let $\underline{u} \in \mathbb{R}^2$ be nonzero vector, if $\underline{x} \in \mathbb{R}^2$, **projection** of $\underline{x}$ onto $\underline{u}$ is vector $P_{\underline{u}}(\underline{x})$ with: (1) $P_{\underline{u}}(\underline{x})$ is a multiple of $\underline{u}$ and (2) $\underline{x} - P_{\underline{u}}(\underline{x})$ is perpendicular to $\underline{u}$. Projection is a linear transformation, and so is rotation of a point an angle about the origin.
4. Let M be $n\times m$ matrix, then function $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(\underline{x}) = M\underline{x}$ for every $\underline{x} \in \mathbb{R}^m$ is a linear transformation. (Every Matrix Defines a Linear Transformation)
5. Let $V = \{\underline{v}_1, ..., \underline{v}_n\}$ be basis for $\mathbb{R}^n$. If $\underline{x} \in \mathbb{R}^n$ then $\underline{x} = \alpha_1\underline{v}_1 + ... + \alpha_n\underline{v}_n$, which is a unique expansion, denoting **coordinates** of $\underline{x}$ with respect to basis V. **Identity** transformation $I(\underline{x}) = \underline{x}$ for all $\underline{x} \in \mathbb{R}^n$
6. Matrix of **linear transformation**: for basis V, W, find the image of $T : V \to W$, as a linear combination of vectors in W, and put the coefficients as columns in the resulting in a **transition** matrix A. When **changing the basis**, of $\underline{u}$ from V to W, multiply matrix A by coordinates of $\underline{u}$ with respect to V, and the result will be the coordinates of $\underline{u}$ with respect to W.
7. Let square matrix $A_{n\times n}$ and $\underline{r}$ a non-zero column vector. $A\underline{r} = \lambda\underline{r}$, where $\lambda \in \mathbb{R}$ is called **Eigenvalue** and $\underline{r}$ is an **Eigenvector**. (When pre-multiplied by A, vector $\underline{r}$ doesn't change direction). A number λ is an eigenvalue of the matrix A iff $|A - \lambda I| = 0$ (*characteristic equation*) of A, and is a polynomial of degree n. Eigenvalues may be complex.
8. For $A_{n\times n}$, if $V^{-1}AV = D = \text{diag}(\lambda_1, ...\lambda_n)$, for $V_{n\times n}$ with columns $[\underline{v}_1, ..., \underline{v}_n]$, then those $\underline{v}_i$ are eigenvectors of A and λ_i are corresponding eigenvalues.
   To **find P,D:** solve characteristic equation, find eigenvalues, express vectors as variable times a vector of coefficients, those coefficients will become columns in P, and D is a diagonal matrix of those eigenvalues.

Sequences

1. Sequences are infinite lists of numbers defined by a formula of the n^{th} term, like $(2^n) = (1, 2, 4, 8...)$ or recursively: $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ (Fibonacci)

2. Sequence (a_n) of real numbers **converges** to a **limit** $l \in \mathbb{R}$ if $\forall \epsilon > 0. \exists N \ : \ |a_n - l| < \epsilon$ for all $n > N$. If it converges, then $\lim_{n\to\infty} a_n = l$ or $a_n \to l$. All constant sequences converge.

3. **Combination Rules for Convergent Sequences**
Convergent sequences $a_n \to \alpha$, $b_n \to \beta$, $c_n \to \gamma$, then:

Sum rule	$a_n + b_n \to \alpha + \beta$
Scalar multiple rule	$\lambda a_n \to \lambda\alpha \quad (\text{for } \lambda \in \mathbb{R})$
Product rule	$a_n b_n \to \alpha\beta$
Reciprocal rule	$1/a_n \to 1/\alpha \quad (\alpha \neq 0)$
Quotient rule	$b_n/a_n \to \beta/\alpha \quad (\alpha \neq 0)$
Hybrid rule	$b_n c_n/a_n \to \beta\gamma/\alpha \quad (\alpha \neq 0)$

4. Sequence a_n is bounded **above** if $\exists U. \forall n : a_n \leq U$, bounded **below** $\exists L. \forall n : a_n \geq L$, and is **bounded** if such U, L exist. A sequence a_n is **increasing** if $\forall n : a_{n+1} \geq a_n$, and **decreasing** if $\forall n : a_{n+1} \leq a_n$. **Subsequence** of a sequence is obtained by deleting some terms.

5. **Basic properties of convergent sequences**
1) A convergent sequence has a **unique limit**
2) $a_n \to l$, then every subseq. of (a_n) also conv. to l
3) If $a_n \to l$ then $|a_n| \to l$. $\quad \downarrow$ **Squeeze rule** $\downarrow$
4) $a_n \to l \wedge b_n \to l$ and $\forall n : a_n \leq c_n \leq b_n$, then $c_n \to l$
5) Conv. seq. is bounded: $\exists B > 0. \forall n : -B \leq a_n \leq B$
6) Any increasing sequence, bounded above and decreasing seq. bounded below, converges.

6. Sequence (a_n) **diverges to infinity** if $\forall K \in \mathbb{R}. \exists N | n > N \Rightarrow a_n > K$. If it does diverge, we write $a_n \to \infty$. A non-convergent, non-divergent sequence **oscillates**.

7. **Basic Convergent Sequences**

(1) $\lim_{n\to\infty} \frac{1}{n^p} = 0$	for any $p > 0$
(2) $\lim_{n\to\infty} c^n = 0$	for any c with $|c| < 1$
(3) $\lim_{n\to\infty} c^{1/n} = 1$	for any $c > 0$
(4) $\lim_{n\to\infty} n^p c^n = 0$	for $p > 0$ and $|c| < 1$
(5) $\lim_{n\to\infty} \frac{c^n}{n!} = 0$	for any $c \in \mathbb{R}$
(6) $\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c$	for any $c \in \mathbb{R}$

Recurrences

1. Recurrence is a rule which defines each term of a sequence using the preceding terms.

2. **Linear recurrences** with constant coefficients of the form: $x_n + a_1 x_{n-1} + \ldots + a_k x_{n-k} = f(n)$, ($a$: constant, f: function). If values of first k terms are given, then it's a unique sequence (x_n). **Homogeneous** recurrence: $\forall n : f(n) = 0$.

3. General solution of recurrence $x_n + a x_{n-1} + b x_{n-2} = 0$ when $b = 0$ is: $\underline{x_n = \lambda^n A}$. And its **Auxiliary** equation is: $\underline{\lambda^2 + a\lambda + b = 0}$. Let λ_1, λ_2 be its roots.
If $\lambda_1 \neq \lambda_2$ then $x_n = A\lambda_1^n + B\lambda_2^n$
If $\lambda_1 = \lambda_2$ then $x_n = A\lambda_1^n + Bn\lambda_2^n$
Use first 2 terms to find A, B by substitution.

4. **Non-homogeneous** recurrence:
(1) Find general solution $x_n = h_n$ of homogeneous recurrence (=0),
(2) Find *any* particular solution $x_n = p_n$ of the original recurrence (= $f(n)$ (replace x_n with a polynomial of degree of $f(n)$, like $x_n = Cn + D$)
(3) General solution will be $x_n = h_n + p_n$.

Series

1. **Series** $\sum a_n$. is a pair of sequences consisting of $(a_n) \to$ sequence of terms, and $(s_n) = a_0 + \ldots + a_n \to$ sequence of partial sums.

2. If (s_n) of partial sums converges to s, then series $\sum a_n$ converges to the sum s: $\sum_{n=0}^{\infty} a_n = s$, diverges otherwise. Try to simplify the expression for a_n, then find s_n, and usually subtract one from another to end up with an easy solution, and take a limit of that expression to find the answer.

3. **Sum Rule**:
   $\sum a_n \to \alpha$ and $\sum b_n \to \beta$ then $\sum (a_n + b_n) \to (\alpha + \beta)$
   Multiple Rule:
   if $\sum a_n \to \alpha$ and $\lambda \in \mathbb{R}$, then $\sum \lambda a_n \to \lambda \alpha$
   Other rules:
   If series $\sum a_n$ converges, then sequence $(a_n) \to 0$
   If series $\sum |a_n|$ converges, then $\sum a_n$ also converges.

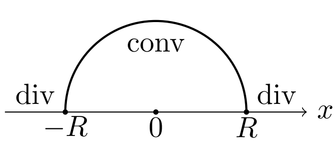
4. **Comparison Test**: Suppose $\forall n : 0 \le a_n \le b_n$, then
   if $\sum b_n$ converges then so does $\sum a_n$
   if $\sum a_n$ diverges then so does $\sum b_n$

5. **Ratio Test**: if $|\frac{a_{n+1}}{a_n}| \to L$ then
   if $0 \le L < 1$ then $\sum a_n$ converges
   if $L > 1$ or $L = \infty$ then series $\sum a_n$ diverges
   if $L = 1$ then test is inconclusive.

6. **Basic Convergent Series**
   (1) $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for any r with $|r| < 1$.
   (2)The series $\sum \frac{1}{n^k}$ converges for any $k > 1$.
   (3)The series $\sum n^k r^n$ converges for $k > 0$ and $|r| < 1$.
   (4) $\sum_{n=0}^{\infty} \frac{c^n}{n!} = e^c$ for any $c \in \mathbb{R}$.
   However, $\sum \frac{1}{n^k}$ diverges $\forall k \le 1$

7. **Power Series** of form $\sum a_n x^n$ for $n \ge 0$.
   Lemma: If $\sum a_n R^n$ converges for some $R \ge 0$, then $\sum a_n x^n$ converges $\forall |x| < R$

8. $R \ge 0$ is **radius of convergence** of $\sum a_n x^n$ if this power series converges $\forall |x| < R$ and diverges for $\forall |x| > R$. If series converges $\forall x$ then $R = \infty$. Radius of convergence defines function $f : (-R, R) \to \mathbb{R}$ given by $f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \forall x \in (-R, R)$. Usually find it using ratio test



9. **Basic Properties of Power Series**
   Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with radii $x \in (-R_1, R_2), x \in (-R_2, R_2)$ where $R_1, R_2 > 0$ and $R = min(R_1, R_2)$. Then:
   Equality rule:
   • If $f(x) = g(x)$ $\forall x \in (-R, R)$ then $a_n = b_n \forall n$
   Sum rule:
   • $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$
   Multiple rule:
   • $\lambda f(x) = \sum_{n=0}^{\infty} \lambda a_n x^n \quad \forall \lambda \in \mathbb{R}$
   Product rule:
   • $f(x) g(x) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + ... + a_n b_0) x^n$

10. **General Binomial Theorem**
    $\forall q \in \mathbb{R} : (1+x)^q = \sum_{n=0}^{\infty} \binom{q}{n} x^n, \quad x \in (-1, 1)$
    where $\binom{q}{n} = \frac{q(q-1)...(q-(n-1))}{n!}$

11. **Partial Fractions** (deg of numerator > denominator)
    $\frac{cx+d}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \Rightarrow cx + d = A(x-b) + B(x-a)$
    Hence, $A = \frac{cx+d}{x-b}\Big|_{x=a}$, $B = \frac{cx+d}{x-a}\Big|_{x=b}$ **(cover up rule)** for any number of factors in denominator with **no repeated factor**. **UNDERSTAND THIS!** ***note 17***

Decimal Representation of $\mathbb{R}$

1. General form of **terminating decimal**: $.a_1 a_2 \cdots a_n$

2. General form of **repeating decimal**: $.a_1 a_2 \cdots a_m \dot{b}_1 \dot{b}_2 \cdots \dot{b}_n$, where b is a repeating sequence of characters. To express it as a **rational** number:
   $0.59\dot{1}0\dot{2} = \frac{59}{100} + \frac{102}{10^5} + \frac{102}{10^8} + ... = \frac{59}{100} + \frac{102}{10^5}(1 + \frac{1}{10^3} + \frac{1}{10^6} + ...) = \frac{59}{100} + \frac{102}{10^5}\left(\frac{1}{1-1/10^3}\right)$ By formula of $\sum_{n \ge 0} r^n = \frac{1}{1-r}$.
   Giving: $\frac{59}{100} + \frac{102}{100}\frac{1}{999}$ = **59043/99900**

3. $\forall x \in \mathbb{R} : a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} ... < x < a_0 + \frac{a_1}{10} + \frac{a_2+1}{10^2} ...$, both converging to x, hence approximating any real number.

4. **Second Order ODE particular solutions for** $f(x)$
   $f(x) = e^{\alpha x}$, then form for a particular solution y:
   1. $y = Ae^{\alpha x}$ if α is not a root of the auxiliary equation
   2. $y = Axe^{\alpha x}$ if α is a non-repeated root
   3. $y = Ax^2 e^{\alpha x}$ if α is a repeated root

   $f(x)$ is polynomial of degree n
   1. pol of deg n if 0 is not a root of the Aux. equation
   2. pol of deg $n+1$ if 0 is a non-repeated root
   3. pol of deg $n+2$ if 0 is a repeated root

   $f(x) = A\cos\alpha x + B\sin\alpha x$
   2. $y = C\cos\alpha x + D\sin\alpha x$ if $i\alpha$ is not root of Aux. eq.
   3. $y = x(C\cos\alpha x + D\sin\alpha x)$ otherwise

Limits and Continuity

1. $\lim_{x\to a} f(x) = l$ means: for every sequence (x_n) in some open interval I of $\mathbb{R}$, $a \in I$ with $x_n \to a, x_n \neq a$, for all n, the sequence $(f(x_n))$ converges to l. Note that $\lim_{x\to a^\pm} f(x) = l$ denotes a **limit** from left/right.

2. **Floor** function $\lfloor 3.7 \rfloor = 3$, **Ceiling** function: $\lceil 3.7 \rceil = 4$

$$\lim_{x\to k^-} \lfloor x \rfloor = k-1, \qquad \lim_{x\to k^+} \lfloor x \rfloor = k,$$
$$\lim_{x\to k^-} \lceil x \rceil = k, \qquad \lim_{x\to k^+} \lceil x \rceil = k+1.$$

3. **Combination rules for limits**
If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ then
sum rule $\lim_{x\to a}(f(x)+g(x)) = l+m$
multiple rule $\lim_{x\to a} \lambda f(x) = \lambda l \quad (\lambda \in \mathbb{R})$
product rule $\lim_{x\to a} f(x)g(x) = lm$
quotient rule $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}$ provided $m \neq 0$.
Squeeze rule for limits
If $f(x) \leq g(x) \leq h(x)$ for $x \neq a, \lim_{x\to a} f(x) = l$ and $\lim_{x\to a} h(x) = l$, then $\lim_{x\to a} g(x) = l$.

4. **Continuity**: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. f is **continuous** at a point $a \in D$ if $lim_{x\to a} f(x)$ exists and equals $f(a)$. And $f : D \to \mathbb{R}$. Includes: polynomials, rational functions, modulus, n^{th} root with $n \geq 1 \in \mathbb{Z}$, trigonometrics, exponents, functions def. by power series

5. If f, g continuous at a then so are:
1)$f+g$; 2)$\lambda f (\lambda \in \mathbb{R})$; 3)$fg$; 4)$\frac{f}{g}$; 5)If f cont. at a and g cont. at $f(a)$ then $g \circ f$ is cont. at a

6. Value of $\lim_{x\to a} f(x)$ doesn't depend of $f(a)$, so **can exist** when $f(a)$ doesn't.

7. **Intermediate Value Theorem**: If $f : [a,b] \to \mathbb{R}$ cont. and $f(a), f(b)$ have opposite signs, then $\exists c \in (a,b) | f(c) = 0$. (The function will have to cross x-axis)

8. **Extreme Value Theorem**: If $f : [a,b] \to \mathbb{R}$ cont. then $\exists m, M \in [a,b]. \forall x \in [a,b] | f(m) \leq f(x) \leq f(M)$. (Continous function defined on a closed bounded interval has a minimum and a maximum points). **Minimum** point would be: $\forall x \in [a,b] | f(x) \geq f(m)$, and **maximum**: $f(x) \leq f(m)$.

Integration

1. **Integration** is defined as the area of the region bounded by $m_r = \texttt{glb}$, $M_r = \texttt{lub}$ of $\{f(x) | x_{r-1} \leq x \leq x_r\}$, so $m_r \leq f(x) \leq M_r \forall x_{r-1} \leq x \leq x_r$, so the area between x_{r-1}, x_r is between $(x_r - x_{r-1})m_r$ and $(x_r - x_{r-1})M_r$. To find area between a, b: sum contributions from all such sub-intervals.

2. For each patition $P = \{x_0, ..., x_n\} of [a,b]$ def. lower sum $L(f,P) = \sum_{r=1}^{n} (x_r - x_{r-1})m_r$ and uppoer sum $U(f,P)$: same but M_r. If there is a unique number $L(f,P) \leq \mathbf{A} \leq U(f,P)$ then f is integrable over $[a,b]$ and A is the **definite integral** (area under the graph), denoted by $A = \int_b^a f(x)dx$.

3. **Properties of definite integrals**:
Sum rule: $\int_a^b ((f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
Multiple rule: $\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$
Transitivity?: $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
Get ratioed: $f(x) \leq g(x) \to \int_a^b f(x)dx \leq \int_a^b g(x)dx$

4. **First Fundamental Theorem of Calculus**
$f : [a,b] \to \mathbb{R}$ integrable, $F : [a,b] \to \mathbb{R}$ such that: $F(x) = \int_a^x f(t)dt$. If f continuous at $c \in (a,b)$ then F differentiable at c and $F'(c) = f(c)$

5. **Second Fundamental Theorem of Calculus**
$f : [a,b] \to \mathbb{R}$ continuous, F differentiable with $F' = f$, then $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$.
Alternatively: $\int f(x)dx = F(x) + c$ for constant c.

6. **DON'T FORGET THE CONSTANT c**

Second Order ODEs

1. **Second order ODE**: $ay'' + by' + cy = f(x)$ where a, b, c are constants. When $f(x) = 0$, then the equation is **Homogeneous**: $y = P(x)$ is a particular solution, $y = H(x)$ is general solution at $f(x) = 0$, then $\mathbf{y} = \mathbf{H}(\mathbf{x}) + \mathbf{P}(\mathbf{x})$ is general sol. of $ay'' + by' + cy = f(x)$

2. **Auxiliary equation** has form $a\lambda^2 + b\lambda + c = 0$
1. $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
2. $\lambda_1 = \lambda_2 \in \mathbb{R}$, then $y = (A + Bx)e^{\lambda x}$
3. $\lambda_1, \lambda_2 \in \mathbb{C}$, then $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$, where $\lambda_1, \lambda_2 = \alpha \pm i\beta$ so $y_1 = e^{\alpha x}\cos\beta x$, $y_2 = e^{\alpha x}\sin\beta x$

Differentiation

1. Real function $f : A \to B$ is **differentiable** at $a \in \mathbb{A}$ if $\exists \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \equiv \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and differentiable if true for all $a \in A$. Also denoted as: $f'(x) = \frac{d}{dx} f(x)$ in Leibniz notation

2. **Theorem**: If f is differentiable at a then f is continuous at a.

3. **Combination Rules for Derivatives**
Sum rule: $(f+g)' = f' + g'$
Multiple rule: $(\lambda f)' = \lambda f'$
Product rule: $(fg)' = fg' + f'g$
Quotient rule: $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$
Chain rule $(g \circ f)'(x) = g'(f(x))f'(x) \equiv \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$
They are all differentiable if f, g are. Also, trig rules: $(\sin x)' = \cos x \quad (\cos x)' = -\sin x \quad (\tan x)' = \frac{1}{\cos^2 x}$

4. If $\sum(a_n x^n)$ is a **power series** with radius of conv. R, then function $f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (-R < x < R)$, then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

5. **Exponential** function can be defined as the sum of a power series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \forall x \in \mathbb{R}$. Hence, $\frac{d}{dx} e^x = e^x$

6. **Partial Derivative** of a function $f(x,y)$ of independent variables x, y comes from differentiating $f(x,y)$ with respect to one of the variables, while holding the other constant. So, if $f(x,y) = x^2 + 8y$ then $\frac{\partial f(x,y)}{\partial x} = 2x$, and $\frac{\partial f(x,y)}{\partial y} = 8$. Often write $\frac{\partial f}{\partial x} \equiv f_x$

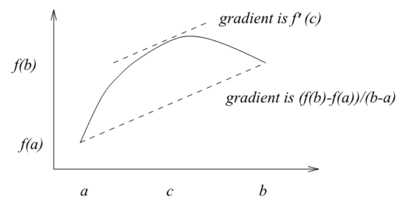
7. **Turning Points**: If $\forall x : f(x) \le f(a)$ then f has a **local maximum** at a, and if $\forall x : f(x) \ge f(a)$ - a **local minimum** at a. Both are called **turning points** of f.

8. **Turning Point Theorem**: If a differentiable function f has a turning point at a then $f'(a) = 0$.

9. Point a where $f'(a) = 0$ is a **stationary point** of f, and it need not be a turning point. If it's not, then it's a **point of inflection**. To locate **maxima, minima**: consider only stationary points and the end points.

10. **Rolle's Theorem**: If $f : [a,b] \to \mathbb{R}$ is continuous, differentiable on (a,b) and $f(a) = f(b)$, then $\exists c \in (a,b) | f'(c) = 0$. (There has to be a turning point between them if the function returns to the same value)

11. **Mean Value Theorem**: If f cont. on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b) | f'(c) = \frac{f(b)-f(a)}{b-a}$. (its gradient is parallel to the line joining a, b)



12. If f cont. on $[a,b]$, diff. on (a,b), then:
1. $f'(x) = 0 \ \forall x \in (a,b) \to$ f constant on $[a,b]$, or $f'(x) > 0 \to$ increasing, $f'(x) < 0 \to$ decreasing.
2. Second Derivative Test: Suppose $f'(c) = 0$, then if $f''(c) > 0$ then f has a local minimum at c, if $f''(c) < 0$, then a local maximum.

13. Function F is an **indefinite integral** of f if $F' = f$. Any two indefinite integrals of f can differ only by a constant $c = G(x) - F(x)$

14. **L'Hôpital's Rule**: Suppose $f(x) = g(x) = 0$, then if $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists then so does $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}$, **NOT** $(\frac{f(x)}{g(x)})'$. If L'Hôpital's doesn't work on the first try, do it again with $f''(x), g''(x)$

15. **Implicit functions** of x are defined by an equation relating x and some other variable. Differentiate as normal, when deriving the other variable, use chain rule (insert $\frac{dy}{dx}$). E.g. $x^2 + y^2 = 5$, find f'_x, so $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

16. Function $f : A \to B$ is
Surjective: $\forall y \in B. \exists x \in A | f(x) = y$,
Injective: $\forall x, y \in A. f(x) = f(y) \to x = y$ and
Bijective: $\forall y \in B. \exists! x \in A | f(x) = y$, or iff it has inverse function $f^{-1} : B \to A$ s.t. $f^{-1}(y) = x \equiv f(x) = y$

17. Can make f bijective by considering subsets of domain/subdomain, hence if $f : A \to B$ is inj. and has range C then $f : A \to C$ bij. hence there is inverse $f^{-1} : C \to A$. **Continuity of inverse functions**: If $f : [a,b] \to \mathbb{R}$ is **continuous** injective function, then inverse $f^{-1} : C \to [a,b]$ is also cont.

18. **Differentiation of inverse functions**: Let $f : [a,b] \to \mathbb{R}$ a continuous function, if f differentiable on (a,b) and is strictly increasing/decreasing then f has an inverse f^{-1} which is differentiable. If $y = f(x)$ then $(f^{-1})' = \frac{1}{f'(x)} \equiv \frac{dy}{dx} = \frac{1}{dy/dx}$

19. **Derivatives of inverses of trigonometric** f:
arcsin' $= \frac{1}{\sqrt{1-x^2}}$ arccos' $= \frac{-1}{\sqrt{1-x^2}}$ arctan' $= \frac{1}{1+x^2}$
write $f(x) = y$, find derivative $\frac{dx}{dy}$, then find $\frac{dy}{dx}$

Logarithms and Exponents

1. **Logarithm**: $\log x = \int_1^x \frac{1}{t} dt$, which exists since integrand ($\frac{1}{t}$) continuous on interval between $(1, x)$ $\forall x > 0$
2. **Properties of logarithm**:
   1. $\log(1) = 0$
   2. Log is **strictly increasing**: $x < y \rightarrow \log x < \log y$
   3. Log is differentiable $\frac{d}{dx} \log x = \frac{1}{x}$ $\forall x > 0$
   4. $\log(xy) = \log(x) + \log(y)$ $\forall x, y > 0$
   5. $\log(x/y) = \log x - \log y$
   6. Function $\log : (0, \infty) \rightarrow \mathbb{R}$ is bijective.
3. **Exponential**: Since $\log : (0, \infty) \rightarrow \mathbb{R}$ bijective, it has inverse function $\exp : \mathbb{R} \rightarrow (0, \infty)$. So $y = \exp(x) \equiv x = \log y \quad (x \in \mathbb{R}, y > 0)$
4. **Properties of exponents**:
   1. $\exp(x + y) = \exp(x) \exp(y)$
   2. exp is differentiable $\frac{d}{dx} \exp(x) = \exp(x),\ \exp(0) = 1$
   3. $\exp(x) = \lim_{n\rightarrow\infty} \left(1 + \frac{x}{n}\right)^n$
   4. $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
5. $\forall x \in \mathbb{R}$ define $e^x = \exp(x)$, then $e^{x+y} = e^x e^y$ $\forall x, y \in \mathbb{R}$
6. For $a > 0, x \in \mathbb{R}$, $a^x \stackrel{\text{def}}{=} e^{x \log a}$, then $\forall a, b > 0, x, y \in \mathbb{R}$: **1**.$(ab)^x = a^x b^x$ **2**.$a^x a^y = a^{x+y}$ **3**.$(a^x)^y = a^{xy} = (a^y)^x$
7. **Change of base**: $\log_b x = \frac{\log x}{\log b}$ $(b \neq 1)$, hence: $y = \log_b x \ \equiv\ \log x = y \log b \ \equiv\ x = b^y$

Taylor's theorem

1. Let f be an $(n+1)$-times differentiable function on an open interval containing points a and x. Then:
$$f(x) = f(a) + f'(a)(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$
for c between a, x.
2. $T_n(x) = a_0 + ... + a_n(x-a)^n$ where $a_i = \frac{f^{(i)}(a)}{i!}$ is called the **Taylor polynomial** of degree n of f at a. It's a polynomial which approximates function f in some interval containing a, and the error in approx. is given by remainder term $R_n(x)$.
3. Since $\lim_{n\rightarrow\infty} R_n(x) = 0$, better approximation, called **Taylor series** for f:
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
4. When $n = 0$, Taylor's theorem reduces to Mean Value Theorem, which is a consequence of Rolle's theorem.
5. When first $n - 1$ derivatives vanish at a, then by Taylor's theorem: $f(x) - f(a) = R_{n-1}(x) = \frac{f^{(n)}(c)}{n!}(x-a)^n$, which helps prove the following statement (6).
6. **$\mathbf{n^{th}}$ derivative test for nature of station. points** Suppose f has stationary point at a, and $f'(a) = \cdots = f^{(n-1)}(a) = 0$, but $f^{(n)} \neq 0$, so if $f^{(n)}$ continuous, then
   1. n even, $f^{(n)}(a) > 0$: f has local **minimum** at a.
   2. n even, $f^{(n)}(a) < 0$: f has local **maximum** at a.
   3. n odd: f has point of **inflection** at a.
7. **Maclaurin Series**: Take $a = 0$ in Taylor's theorem results in:
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n, \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$
with $f^{(0)}(0) = f(0)$, so to find those series, just replace f with your function: $e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

First Order ODEs

1. **Ordinary differential equation** (ODE) is an equation containing derivatives of a a function of a single variable: $y' = 4x \quad y'' + 4y = x$ etc. **Order** of diff. eq. is that of the highest derivative it contains, so above equations have order of 1 (y') and 2 (y'').
2. **General solution** contains n arbitrary constants, **particular** solution doesn't contain any. $y' = 4x \Rightarrow \int y'dy = \int 4xdx \Rightarrow \underline{y = 2x^2 + c}$ (general sol.), $\underline{y = 2x^2 + 2 \text{ (particular sol.)}}$ if $y = 4$ when $x = 1$
3. **Separable Equations** is ODE of form: $\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{1}{g(y)}\frac{dy}{dx} = f(x) \Rightarrow \int \frac{1}{g(y)}dy = \int f(x)dx$, solving which will give a solution. **Remember the constant c**, **can separate** $\frac{dy}{dx}$ e.g. $\frac{dy}{dx} = 1 \Rightarrow dy = dx$
4. **Homogeneous Equations** is ODE of form $\frac{dy}{dx} = f(y/x)$. Let $v = y/x \Rightarrow y = xv$, then $\frac{d}{dx}(xv) = f(v) \Rightarrow x\frac{dv}{dx} + v = f(v) \Rightarrow \frac{dv}{dx} = \frac{f(v)-v}{x}$, which is separable. **Remember** $\mathbf{\frac{d}{dx}(xv) = (xv)' = v + x\frac{dv}{dx}}$ (chain rule).
5. **Linear Equations** is ODE of form $\frac{dy}{dx} + P(x)y = Q(x)$ Take $Q(x) = 0$, which gives $y = e^{-\int P(x)dx}$, let **Integrating factor** $\underline{I(x) = e^{\int P(x)dx}}$ s.t. $yI(x) = 1$. Differential $\frac{d}{dx}I(x) = I(x)P(x)$. Multiply both sides of original equation by $I(x)$ to get: $yI(x) = \int Q(x)I(x)$ (**general solution**), or substitute known values to get **particular** solution (get rid of c).