Lecture Notes

CS130

Logic

General Logic

1. $\neg$ negation (NOT)
2. $\wedge$ conjunction (AND)
3. $\vee$ disjunction (OR)
4. $\oplus$ exclusive or (XOR)
5. $\in$ membership
6. $\rightarrow$ implication
7. $\equiv$ or $\leftrightarrow$ equivalence
8. $\top$ "top" always returns T
9. $\bot$ "bottom" always returns F
10. $\exists$ existential quantifier ("exists")
11. $\forall$ universal quantifier ("forall")

$A \rightarrow B$: A is sufficient for B, B is necessary for A

$A \leftrightarrow B$: A/B is necessary and sufficient for B/A

Theorems & Rules

1. $A \rightarrow B \equiv \neg B \rightarrow \neg A \equiv \neg A \vee B$
2. $A \leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$
3. $A \wedge B \equiv \neg(\neg A \vee \neg B)$
4. $A \vee B \equiv \neg(\neg A \wedge \neg B)$

Property	**Statement**
Associativity	$(x \vee y) \vee z \equiv x \vee (y \vee z)$ $(x \wedge y) \wedge z \equiv x \wedge (y \wedge z)$
Commutativity	$x \vee y \equiv y \vee x$ $x \wedge y \equiv y \wedge x$
Identity Laws	$x \vee F \equiv x$ $x \wedge T \equiv x$
Idempotence	$x \vee x \equiv x$ $x \wedge x \equiv x$
De Morgan's Laws	$\neg(x \vee y) \equiv \neg x \wedge \neg y$ $\neg(x \wedge y) \equiv \neg x \vee \neg y$
Excluded Middle	$x \vee \neg x \equiv T$ $x \wedge \neg x \equiv F$
Doub. Neg.	$\neg\neg x \equiv x$
Annihilation	$x \wedge F \equiv F$ $x \vee T \equiv T$
Absorption	$x \vee (x \wedge y) \equiv x$ $x \wedge (x \vee y) \equiv x$
Distributivity	$x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$ $x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$

Predicates

1. Predicate is a function producing truth value.
2. Proposition is a thing with attached truth value.
3. Atomic proposition: F, T, [1+1=2].
4. Compound: atomic prop.'s connected by operators.
5. Totology is a composition that is always true
6. $\exists!$ - there exists exactly 1

Express finite set predicates using ($\wedge$) and ($\vee$)

1. $\forall x \in S : P(x) \equiv P(a_1) \wedge ... \wedge P(a_n)$
2. $\exists x \in S : P(x) \equiv P(a_1) \vee ... \vee P(a_n)$

De Morgan's laws on predicates

3. $\neg \forall x : P(x) \equiv \exists x : \neg P(x)$
4. $\neg \exists x : P(x) \equiv \forall x : \neg P(x)$

When Q contains x as a free variable

5. $(\forall x : P(x)) \wedge (\exists x : Q(x)) \equiv \forall x : (P(x) \wedge Q(x))$
6. $(\exists x : P(x)) \vee (\exists x : Q(x)) \equiv \exists x : (P(x) \vee Q(x))$

When Q doesn't contain x as a free variable

7. $(\forall x : P(x)) \wedge Q \equiv \forall x : (P(x) \wedge Q)$
8. $(\exists x : P(x)) \vee Q \equiv \exists x : (P(x) \vee Q)$
9. $(\forall x : P(x)) \vee Q \equiv \forall x : (P(x) \vee Q)$
10. $(\exists x : P(x)) \wedge Q \equiv \exists x : (P(x) \wedge Q)$
11. $(\forall x : P(x)) \rightarrow Q \equiv \forall x : (P(x) \rightarrow Q)$
12. $(\exists x : P(x)) \rightarrow Q \equiv \exists x : (P(x) \rightarrow Q)$
13. $Q \rightarrow (\forall x : P(x)) \equiv \forall x : (Q \rightarrow P(x))$
14. $Q \rightarrow (\exists x : P(x)) \equiv \exists x : (Q \rightarrow P(x))$
15. $(\forall x : P(x)) \equiv Q \equiv \forall x : (P(x) \equiv Q)$
16. $(\exists x : P(x)) \equiv Q \equiv \exists x : (P(x) \equiv Q)$

Other rules:

17. $\neg \forall x . P(x) \equiv \exists x . \neg P(x)$
18. $\neg \exists x . P(x) \equiv \forall x . \neg P(x)$

General Set Theory

Set is a maths object with notion of membership.

1. $0 \in \mathbb{N}$
2. {Elem} singleton
3. $\varnothing \equiv \{\}$ empty set
4. $\subseteq$ subset or equal
5. $\subsetneq$ or $\subset$ "proper subset" (not equal)
6. $\varnothing \subseteq S = T$ but $\varnothing \subset S = F$ for any set S
7. Set builder notation: $\{x \in S | P(x)\}$
8. $|A|$ set cardinality(num unique top lvl elems)
9. $|\mathbb{N}|$ = undefined
10. Can't have a set of all sets
11. $\mathcal{P}(S)$ or 2^S Powerset (set of all subsets of S)
12. Powerset of finite set is finite, infinite for infinite
13. A, B are disjoint sets if $A \cap B = \varnothing$
14. $\cup$ union, $\cap$ intersection, $\setminus$ difference, $\times$ Cart. prod.
15. Chosen universal set has all "universe" elements
16. Set complement: $\overline{A} = S \setminus A$; S is universal set
17. If A infinite, $B \neq \varnothing$, then $A \times B$ infinite.
18. $S \circ R$ set composition
19. $A \triangle B$ symmetric difference
20. Equinumerous $A \cong B$ if $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ bij. so $|A| = |B|$
21. $R_{\cong} : \mathcal{P}(S) \leftrightarrow \mathcal{P}(S)$ is an equivalence relation
22. Set A is finite if $\exists n | A \cong \mathbb{N}_n$. This n is unique.
23. Infinite set A is countable if $A \cong \mathbb{N}$
24. A(countable)$/R_{\sim}$ is either finite or countable.
25. $\mathbb{Q}$ is dense $(\forall a, c \in \mathbb{Q} | \exists b$(inf. num.)$\in \mathbb{Q}$ s.t. $a < b < c$
26. $\mathbb{Q} = Q_1 \cup Q_2 | \forall x \in Q_1, y \in Q_2 . x < y$ Dedekind cuts
27. $[0, 1] \cong \mathbb{R}$ is uncountably infinite.

Axioms and Theorems

1. $(A \subseteq B \wedge B \subseteq A) \rightarrow A = B$ *Law of Extensionality*
2. $\exists A = \{x | P(x)\}$ s.t $x \in A$ iff $P(x)$. *Law of Abstraction*
3. $B = \{A | A \notin A\}$ Russel Paradox
4. $A \subseteq B \equiv \forall x : x \in A \rightarrow x \in B$ Subset operator
5. $A \triangle B = (A \cup B) \setminus (A \cap B)$
6. $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$
7. $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$ De Morgan's
8. $f : A \rightarrow B$ injective then $A \cong B'$ for some $B' \subseteq B$
9. For all finite sets A, $\exists n$(unique) s.t. $A \cong \mathbb{N}_n$
10. $\mathbb{N}^+, \mathbb{N}_{\text{even}}, \mathbb{N}^{n \in \mathbb{N}}$(diagonalisation)$, \mathbb{Z}$ are countable.
11. $\forall A | A \ncong \mathcal{P}(A)$(uncount.) Cantor's diagonal argument

Rules and Definitions

1. $\{x \in S | T\} = S$
2. $\{x \in S | F\} = \varnothing$
3. $\mathcal{P}(S) = \{A | A \subseteq S\}$
4. $|\mathcal{P}(S)| = 2^{|S|}$
5. $A \cup B = \{x | x \in A \vee x \in B\}$; $A \subseteq (A \cup B)$
6. $A \cap B = \{x | x \in A \wedge x \in B\} \subseteq A$
7. $A \setminus B = \{x | x \in A \wedge \neg(x \in B)\} \subseteq A$
8. $A \triangle B = \{x | x \in A \oplus x \in B\} = (A \setminus B) \cup (B \setminus A)$
9. $A \times B = \{(a, b) | a \in A, \ b \in B\}$
10. $A \times \varnothing = \varnothing \times A = \varnothing$
11. $A \times B \times C \neq (A \times B) \times C$
12. If $(A \subseteq B) \wedge (B \subseteq C)$ then $A \subseteq C$
13. $A \subseteq B \equiv x \in A \rightarrow x \in B \equiv \neg \exists x \in A . x \notin B$
14. $A \subseteq B \wedge B \subseteq A \rightarrow A = B$
15. $\mathbb{N}_n = \{x \in \mathbb{N} | x < n\}$, so $\mathbb{N}_0 = \varnothing, \mathbb{N}_2 = \{0, 1\}$
16. $\mathbb{Q} = \mathbb{Z}^2 / R_{\sim}$
17. $A \times B \neq B \times A$ if $A \neq B$
18. $A^2 = A \times A$, $\{a, b\}^2 = \{(a, a), (a, b), (b, a), (b, b)\}$
19. $|A \times B| = |A| \cdot |B|$
20. $\varnothing \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

$A \cap S = A$; $A \cup \varnothing = A$ Identity
$A \cap \varnothing = \varnothing$; $A \cup S = S$ Annihilation
$A \cap \overline{A} = \varnothing$; $A \cup \overline{A} = S$ Excluded middle
$A = A \cap (A \cup B) = A \cup (A \cap B)$ Absorption

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1. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
3. $(A \cap B) \times C = (A \times C) \cap (B \times C)$
4. $(A \cup B) \times C = (A \times C) \cup (B \times C)$
5. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$
6. $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$

- $A_i \overset{\text{def}}{=} \{x \in \mathbb{N} | x \leq i\} \ \forall i \in \mathbb{Z}$
- $\bigcup_{i=1}^{n} \overset{\text{def}}{=} A_1 \cup ... \cup A_n \qquad \bigcap_{i=1}^{n} \overset{\text{def}}{=} A_1 \cap ... \cap A_n$

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1. $S \circ R = \{(x, z) | \exists y . (x, y) \in R \wedge (y, z) \in S\}$
2. $x \in A \setminus B \Rightarrow x \in A \cup B, x \notin A \cap B$
3. Reflexive $A \cong A | \text{id}_A : (x \in A) \mapsto x$
4. Symmetric $A \cong B \rightarrow B \cong A$
5. Transitive $A \cong B \wedge B \cong C \rightarrow A \cong C$

General

1. $R_p \subseteq A \times B$ or apb with $a \in A, b \in B$ is a Relation
2. For any operator "$\sim$": $R_\sim = a \sim b$ iff $(a, b) \in R$
3. $A = B$ i.e. $A \times A = A^2$, then say "relation on A"
4. $R^{-1} = \{(b, a) | (a, b) \in R\} \subseteq B \times A$ Inverse relation
5. $R_\perp \equiv \varnothing \subseteq A \times B$ Empty relation
6. $R_\top \equiv A \times B \subseteq A \times B$ Complete relation
7. $R_{=_A} = \{(a, a) | a \in A\}$ Equality relation
8. $R_|$ divisibility relation.
9. $R_{\equiv_n}$ Congruence modulo n relation
10. $R_\sim$ Equivalence relation
11. $[a]_R$ is the equivalence class of $R_\sim$
12. Classes $[a]_{\equiv_n}$ are called residue classes modulo n
13. $(\mathbb{N}, =)$ set of naturals *equipped* with eq. rel. on $\mathbb{N}$
14. For infinite A, $|A/R_\sim|$ may be finite or infinite.
15. Equivalence relations define similarity, equality.
16. Partial order $R_\preceq$ is total if $\forall a, b \in A | a \preceq b \vee b \preceq a$
17. c is an *upper bound* of b if $b \preceq c$. Note, $\forall b | b \preceq b$
18. d is a *lower bound* of a if $d \preceq a$.
19. c is an *common* upper bound of a, b if $a \preceq c \wedge b \preceq c$
20. d is an *common* lower bound of a, b if $d \preceq a \wedge d \preceq b$
21. `lub` least upper bound, `glb` greatest lower bound
22. <u>lattice</u>: $\exists$`glb,lub` for every two elements in $R_\leq$
23. $R_{\preceq_A} : a \in A$ is <u>maximal</u> if $\forall x \in A | (a \preceq x) \to (a = x)$
24. $R_{\preceq_A} : a \in A$ is <u>minimal</u> if $\forall x \in A | (x \preceq a) \to (x = a)$
25. $a \in A$ is *greatest* element if $\forall x \in A : x \preceq a$
26. $a \in A$ is *least* element if $\forall x \in A : a \preceq x$

Properties

Let $R \subseteq S \times S$. Then we say R is:

1. Reflexive $\forall x \in S | (x, x) \in R$, xRx
2. Irreflexive $\forall x \in S | (x, x) \in R$, $\neg xRx$
3. Symmetric $\forall a, b \in S | aRb \leftrightarrow bRa$
4. Transitive $\forall a, b, c \in S | aRb \wedge bRc \to aRc$
5. Antisymmetric $\forall a, b \in S | aRb \wedge bRa \to a = b$
6. Equivalence relation: Refl(1), Symm(2), Transit(3)
7. Partial order: Refl(1), Transit(3), Antisymm(4)

Reflexive: $R_= \in R_p$, Symmetric: $R_{p^{-1}} \in R_p$
Transitive: $R_{p \circ p} \in R_p$, Antisymm: $R_p \cap R_{p^{-1}} \in R_=$

Theorems

1. Equiv. classes of $R_{\sim^A}$ are pairwise disjoint. The union of all equivalence classes is the whole set A.
2. Divisibility relation $R_| : \mathbb{N} \leftrightarrow \mathbb{N}$ is a partial order.

Rules and Definitions

1. $R_=^{-1} \equiv R_=$, $R_\top^{-1} \equiv R_\top$, $R_\perp^{-1} \equiv R_\perp$
2. $R_{p \circ q} \stackrel{\text{def}}{=} \forall (a, b) \in A \times B |\ a(p \circ q)b$
3. $R_{p \circ q} \stackrel{\text{def}}{=} \forall (a, c) \in A \times C |\ \exists b \in B | (apb) \wedge (bqc)$
4. $[a]_R = \{x \in S | aRx\}$ for $R \subseteq S \times S$
5. $[a]_\sim = \{x \in A | x \sim a\}$, a is <u>representative</u> of $[a]_\sim$
6. $A/R_\sim = \{[a]_\sim | a \in A\}$ Quotient set of A w.r.t. $R_\sim$
7. If $|A| = n$, all $|[a]_\sim| = m$, then $|A/R_\sim| = \frac{n}{m}$.
8. $\min(a, b) \equiv a \sqcap b$, $\max(a, b) \equiv a \sqcup b$
9. $\texttt{glb}_\subseteq(A, B) = A \cap B$, $\texttt{lub}_\subseteq(A, B) = A \texttt{lub} B$

$c = \texttt{lub}_\preceq(a, b) \equiv (a \preceq c) \wedge (b \preceq c) \wedge (\forall x \in A | (a \preceq x) \wedge (b \preceq x) \to (c \preceq x))$

$d = \texttt{glb}_\preceq(a, b) \equiv (d \preceq a) \wedge (d \preceq b) \wedge (\forall x \in A | (x \preceq a) \wedge (x \preceq b) \to (x \preceq d))$

Good to Know

1. Antisymmetric isn't opposite of symmetric relation!
2. Relation with $(a, b), (b, a)$ for some, but not all $a, b \in A, a \neq b$ is neither Symmetric nor Antisymm.
3. $R_=$ is both Symmetric, Antisymmetric.
4. If $a \sim b$ means the lines a, b are parallel, then all possible directions are equivalence classes.
5. $R_\sim$ "person a was born on the same day as person b" has 366 equiv. classes. $A/R_\sim$ = set of all birthdays.
6. $\texttt{glb}_|(a, b) = \texttt{gcd}(a, b)$, $\texttt{lub}_|(a, b) = \texttt{lcm}(a, b)$
7. $aRb \neq bRa$
8. $R_|(\neq 0)$, $R_\subseteq$ and $R_\leq$ have guaranteed `lub`, `glb`
9. minimal, maximal elements are "extremes".
10. greatest elem is maximal, least is minimal
11. maximal elem need not be greatest, minimal-least even if both are unique.
12. greatest and least elements are unique.

General

1. There has to be Existence and Uniqueness!
1. $f : A \to B \equiv R_f : A \to B$ s.t. $\forall a \in A | \exists! b \in B . afb$
2. $f : A$(domain) $\to B$(codomain). f maps A into B
3. $f : A \to A$ is a function on the set A
4. $f : A \to B \equiv (a, b) \in R_f \equiv afb \equiv f(a) = b$
5. We say that function f maps a to b.
6. $b = f(a)$ is the image of a, and a is pre-image of b
7. Complete pre-image: $f^{-1}(y) \stackrel{\text{def}}{=} \{x \in X | f(x) = y\}$
8. $(f \circ g)(a) = g(f(a))$ CHECK THIS
9. Range $f(A)$ of $f : A \to B$ includes all results of f in B
10. Bijective $f : A \rightarrowtail\!\!\!\twoheadrightarrow A$ is a "permutation" on set A
11. If $f(f(A)) = A$ then f is an "involution" on set A
12. A function must have a single output for each input.
13. Sign func: $\text{sign}(x) = 1$ if $x > 0, 0$ if $x = 0, -1$ else
14. $\frac{1}{x}$ isn't a function, because not defined if $x = 0$
15. $\pm x$ isn't a func, as it has 2 outputs for each input
16. $f : A \to B$ a function "from" $A \to B$
17. Function = map = mapping = $\mapsto$
18. For $f : A \to B$: $f|_H \stackrel{\text{def}}{=} f \cap ((H \subseteq A) \times B)$

Theorems

1. $f : A \to B, g : B \to C \Rightarrow R_{f \circ g}$ is function $A \to C$
2. $f : A \to B$ bijective iff f^{-1} is also bijective function

Rules and Definitions

1. Identity function: $\text{id}_A : A \to A | \forall a \in A . \text{id}_A(a) = a$
2. Restriction of f on H: $f|_H = \{(a, f(a)) | a \in H\}$ so $f : A \to B$ becomes $f : (H \subseteq A) \to B$.
3. Infinite sequence $(a_0, a_1 ...)$ where $\forall i \in \mathbb{N} : a_i \in A$ is a function $a : \mathbb{N} \to A$
4. Function $f(A) = B$ is surjective. Maps A onto B. Denoted as $f : A \twoheadrightarrow B$.
5. $\forall x, y \in A : (f(x) = f(y)) \to (x = y)$ is injective. Maps A to B one-to-one. Denoted as $f : A \rightarrowtail B$
6. $f : A \to B$ is bijective if surjective and injective. One-to-one correspondence between A, B: $A \rightarrowtail\!\!\!\twoheadrightarrow B$ Has unique pre-image: $\forall b \in B : \exists! a \in A : f(a) = b$
7. Indicator function on $A \in \mathcal{P}(S) : \chi_A : S \to \mathbb{B}$ $\forall x \in S : \chi_A(x) = T$ if $x \in A, F$ if $x \notin A$.
8. Mapping $\chi : A \mapsto \chi_A$ produces a bijection between $\mathcal{B}(S) = \{f | f : S \to \mathbb{B}\}$
9. $f|_H : H \to B$ is undefined on all inputs not in H.
10. $f : A \to B$ then $R_{f^{-1}} : B \leftrightarrow A$ need not be a function

General

1. V: Vertices(Nodes), E: Edges(Pairs of Nodes)
2. Pairs of nodes comprising relation E are called *edges*
3. Two nodes connected by an edge are called *adjacent*
4. $G = (V, E)$: Graph with sets of nodes V and edges E
5. Graph (G) "on" V is an Irreflexive, symmeric relation defined by $E = R_{\rightharpoonup} : V \leftrightarrow V$ (V is any finite set)
6. *Empty graph* has no edges: $(V, \varnothing)$
7. *Complete graph* $K(n)$ contains all possible edges: $K(V) = (V, E)$, where $E = \{(u, v) \in V^2 | u \neq v\}$
8. Graph G is *bipartite* or *two-coloured* if set of nodes can be partitioned into 2 disjoint subsets $V = V_1 \cup V_2$ s.t. every edge in E connects 2 nodes from diff. subsets V_1, V_2 are *colour classes*
9. $K(n)$ can be read as: 'Any graph isomorphic to $K(\mathbb{N}_n)$'
10. $K(V_1, V_2) = (V_1 \cup V_2, (V_1 \times V_2) \cup (V_2 \times V_1))$ is called a complete bipartite graph. $K(m \in \mathbb{N}, n \in \mathbb{N})$ - any graph isomorphic to $K(H, W)$ with m houses, n wells
11. A graph with k colour classes is called k-partite
12. Complete graph has $\frac{n(n-1)}{2}$ edges.
13. Connected graph stays conn. when adding edges
14. Acyclic graph stays acyclic when removing edges
15. Trees are maximal among acyclic graphs

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1. *Eulerian* cycle visits each edge only once.
2. *Hamiltonian* cycle visits each node only once.
3. $V \neq \varnothing, |V|$ finite.
4. Directed graph: ordered pairs: $e = (v, w) \in E$
5. Undirected: unordered $e = \{v, w\} \in E$
6. Self-loops: $e = (v, v)$
7. A graph is *simple* if no loops and multiple edges.
8. edges(e)=v(source) w(destination in dir.)$G \in E$
9. *Multiplicity*: number of edges between 2 nodes.
10. *Adjacent* nodes: Nodes, connected by an edge.
11. *Incident* nodes: Nodes that an edge connects.
12. Self-loops count twice in *Vertex degree*
13. in-deg(v) $\stackrel{\text{def}}{=}$ num. edges where v is destination.
14. out-deg(v) $\stackrel{\text{def}}{=}$ num. edges where v is source.
15. $v \rightarrow w : vw \in E$
16. $v \rightarrow^* w : \exists v \leftrightsquigarrow w$ or w is reachable from v.
17. Graph G is Eulerian if it has an Eulerian cycle.
18. G' is a *subgraph* of $G(G' \subseteq G)$ if $V' \subseteq V, E' \subseteq E$
19. G' *spanning subgraph* of $G(G' \sqsubseteq G)$ if $V' = V, E' \subseteq E$
20. $R_{\subseteq}, R_{\sqsubseteq} : \mathcal{G}(V) \leftrightarrow \mathcal{G}(V)$ are part. orders on $\mathcal{G}(V)$
21. **Tree**: connected, acyclic graph.
22. **Forest**: acyclic graph (not necessarily connected)
23. If deg(v) = 1 in a tree, then v is a *leaf*

Graph Connectivity

1. **Walk** (of len. k) is a sequence $(u, u_1, ...u_{k-1}, v)$ s.t. every two consecutive nodes in the sequence are connected by an edge:$(u \rightharpoonup u_1) \wedge ... \wedge (u_{k-1} \rightharpoonup v)$
2. $u \leftrightsquigarrow v$ "nodes u and v are connected by a walk"
3. **Tour** is a walk that returns to the starting node
4. Nodes u, v in a graph are *connected*, if $\exists u \leftrightsquigarrow v$.
5. A graph is connected if all (u, v) are connected.
6. Connectivity is equivalence relation on the set of all nodes in a graph: $R_{\leftrightsquigarrow} : V \leftrightarrow V$
7. Equivalence classes of $R_{\leftrightsquigarrow}$ are "connected components" of graph G. A graph is connected iff it has only 1 connected component.
8. A walk where all E are distinct is a **Path**: $u \rightsquigarrow v$ $u = u_0 \rightharpoonup ... \rightharpoonup u_k \rightharpoonup v \; \forall i, j \in \mathbb{N}_{k+1} | u_i \neq u_j$
9. **Cycle** is a tour with no edges repeated.
10. A graph without cycles is called *acyclic*
11. $R_{\rightsquigarrow} : V \leftrightarrow V$ is equivalence relation
12. $\deg(v) = |\{u \in V | v \rightharpoonup u\}$ (num adjacent nodes)
13. **Simple Path** is a walk that repeats no vertices.
14. **Simple Cycle** is tour with no vertices repeated except $v_0 = v_n$
15. A graph is planar, if can be embedded in the plane s.t. the lines representing different edges do not cross.
16. Usually want to identify graphs which are "the same up to a renaming of nodes"
17. Graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ are *isomorphic* if bijective func. $f : V_1 \leftrightarrow V_2$, preserving edges exists: $\forall u, v \in V_1 | (u, v) \in E_1 \equiv (f(u), f(v)) \in E_2$, where bijective f is an isomorphism between G_1 and G_2
18. *Subdivision* of graph G is obtained by removing edge $(u, v), (v, u)$ from G, and then adding to G new node $x \notin V$, and new edges $(u, x), (x, v), (x, u), (v, x)$

Theorems

1. $\forall u, v \in V | \exists u \rightsquigarrow v$ iff $\exists u \leftrightsquigarrow v$
2. G has Euler tour iff: G connected, $\forall v \in V : |v|$ even (Euler, Hierholzer)
3. $\sum_{v \in V} \deg(v) = 2 \cdot |E|$ Handshaking Lemma
4. Let $G = (V, E)$ be a tree. Then $|V| = |E| + 1$
5. Every tree with at least one edge has a leaf.
6. Partial order $R_{\sqsubseteq}$ on set of all asyclic graphs on finite set V. Graph $G = (V, E)$ is maximal iff it's a tree.
7. Graph is planar iff it has no subgraph isomorphic to a graph obtained from $K(5)$ or $K(3, 3)$ by a sequence of subdivisions.

General

Direct proof, Induction, Contradiction, Exhaustion, Contra positive, Disproof by Counterexample.

`WLOG`: without loss of generality: presenting specific proof that can be scaled to any example.

`FTSOC`: for the sake of contradiction

`LHS, RHS`: left and right hand sides

Induction

a. Base Case: non self-referential initial statement
b. Inductive step (self-referential def.) i.e. $5 = \text{next}(4)$
c. Completeness statement (domain restriction)
For example: $3 = \text{next}_b(\text{next}_b(\text{next}_b(0_a)) \in \mathbb{N}_c$

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1. Assumption from inductive step is *inductive hypothesis*

Theorems

1) Suppose n is an integer larger than 1 and n is not prime. Then $2n1$ is not prime.

Proofs *continued*

1) $\therefore$, $\Rightarrow$ "therefore"
2) $\because$, $\Leftarrow$ "because"
3) $\Leftrightarrow$ "which is the same as"
4) $\square$ End of proof, $\perp$ Contradiction

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1. Begin each sentence with a word, not a symbol.
2. End each sentence with a period.
3. Separate maths symbols & expressions with words.
4. Don't replace words with symbols.
5. Avoid using unnecessary symbols.
6. Use "we", "us" instead of "I", "you", "me".
7. Use active voice: "dividing RHS by n we get x=3.
8. Explain each new symbol.
9. Don't replace words with "it".
